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Results on Integral Operator of Analytic Function Defined by Fractional Derivatives

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Abstract: In the present study we introduce a new class namely H (A, B, p, δ) of analytic functions defined by fractional derivatives. In our first result we find a class preserving integral operator and in the next result a converse problem of this integral operator is discussed.

Key words: Analytic functions, fractional derivative, integral operator and the class H (A, B, p, δ), univalent functions

INTRODUCTION

In this study, we introduce a new class H (A, B, p, δ) of analytic functions defined by fractional derivative, as defined below:

A function f (z) of T (p) belongs to the class H (A, B, p, δ) if and only if there exists w belonging to the class H (A, B, p, δ) such that:

$$\frac{\Omega_z^{(\delta,p)} f(z)}{\Omega_z^{(\delta-1,p)} f(z)} = \frac{1+AW(Z)}{1+BW(Z)}, \; z \in U \qquad (1)$$

where, $-1 \le A < B \le 1$.
The condition (1.1) is equivalent to:

$$\left| \frac{\Omega_z^{(\delta,p)} f(z) - \Omega_z^{(\delta-1,p)} f(z)}{B\Omega_z^{(\delta,p)} f(z) - A\Omega_z^{(\delta-1,p)} f(z)} \right| < 1 \; z \in U \qquad (2)$$

By giving the specific values to A, B, p and δ in (2), we obtain the following important subclasses studied by various researchers in earlier works:

- For δ = 1, we obtain the class of functions f(z) satisfying the condition

$$\left| \frac{zf'(z) - pf(z)}{Bzf'(z) - Apf(z)} \right| < 1 \; z \in U$$

studied by Goel and Sohi (1980).

- For δ = 1, A = (2α − 1) β, B = β and p = 1, we obtain the class of functions f(z) satisfying the condition:

$$\left| \frac{zf'(z) - f(z)}{zf'(z) - (2a-1)f(z)} \right| < \beta \; z \in U$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$, studied by Gupta and Jain (1976).

- For δ = 1, A = (2α − 1) β, B = β and p = 1, we obtain the class of functions f (z) satisfied the condition:

$$\left| \frac{zf'(z) - f(z)}{zf'(z) - (2a-1)f(z)} \right| < 1 \; z \in U$$

studied by Silverman (1999).

MAIN RESULTS

Recently Sharma and Singh (2010) and Singh and Sharma (2011) obtained a class preserving integral operator for the class H(A, B, f, p, δ) of analytic function in terms of fractional integral operator. Akin to the results given in Singh and Sharma (2011) we find following results, for the same we need a lemma as follows:

Lemma (a): A function f (z) defined by:

$$f(z) = z^p - \sum_{n-1}^{\infty} |a_{p+n}| z^{p+n}$$

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is in the class H (A, B, p, δ) if and only if:

$$\sum_{n=1}^{\infty} \psi(n, p, \delta)\{(1+B)n(B-A)(1+p-\delta)\} \left|a_{p+n}\right| \leq (B-A) \tag{3}$$

Integral operator:

Theorem (a): Let the function f(z) defined by:

$$f(z) = z^{p} - \sum_{n-1}^{\infty} \left|a_{p+n}\right| z^{p+n}$$

is in the class H(A, B, p, δ). Also let c>-p. Then the function F(z) defined by:

$$F(z) = z^{p-1} \int_0^z \left[\frac{f(t)}{t^p}\right] dt$$

is also in the class H (A, B, p, δ).

Proof: By the definitions of f(z) and F(z), it is easily seen that:

$$F(z) = z^{p} - \sum_{n=1}^{\infty} \left|h_{p+n}\right| z^{p+n}$$

where;

$$\left|h_{p+n}\right| = \frac{(c+p)}{(c+p+n)} \left|a_{p+n}\right|$$

Therefore,

$$\sum_{n=1}^{\infty} \psi(n, p, \delta)\{(1+B)n + (B-1)(1+p-\delta)\}\left|h_{p+n}\right|$$

$$= \sum_{n=1}^{\infty} \psi(n, p, \delta)\{(1+B)$$

$$n + (B-A)(1+P-\delta)\} \tfrac{(c+p)}{(c+p+n)} \left|a_{p+n}\right|$$

$$< \sum_{n=1}^{\infty} \psi(n, p, \delta)\{(1+B)(1+P-\delta)\}\left|a_{p+n}\right|$$

≤(B-A), by Lemma (a).

Hence the function F(z) belongs to the class H(A, B, p, δ). Theorem (a) simplifies considerably when we set c = (1-p), and we thus obtain:

Corollary (a): If f (z) defined by above theorem belongs to the class H (A, B, p, δ). Then:

$$F(z) = z^{p-1} \int_0^z \left[\frac{f(t)}{t^p}\right] dt$$

is also belongs to the class H (A, B, p, δ).

In the following theorem, we consider the converse problem of the above theorem.

Theorem (b): Let c>-p. Also let f (z) is in the class H (A, B, p, δ). Then F(z) given in Theorem (a) is p-valent in the disc │z│< R, where:

$$R = \inf_{n \in N} \left[\frac{\{(1+B)n + (B-A)(1+p-\delta)\}(c+p)p\psi(n, p, \delta)}{(B-A)(c+p+n)(p+n)}\right]^{1/n}$$

The result is sharp.

Proof: Let

$$F(z) = z^{p} - \sum_{n-1}^{\infty} \left|h_{p+n}\right| z^{p+n}$$

belongs to the class H (A, B, p, δ). Then from Theorem (a), it follows that:

$$f(z) = \frac{z^{1-c}\{z^c F(z)\}'}{(c+P)} = z^{p} - \sum_{n-1}^{\infty} \left\{\frac{c+p+n}{c+p}\right\} \left|a_{p+n}\right| z^{p+n}$$

In order to establish the required result, it suffices to show that:

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p \text{ for } |z| < R$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{n=1}^{\infty} \frac{(p+n)(c+p+n)}{(c+p)} \left|a_{p+n}\right| \left|z^{n}\right|$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p \text{ , if}$$

$$\sum_{n=1}^{\infty} \frac{(p+n)(c+p+n)}{(c+P)} \left|a_{p+n}\right| \left|z^{n}\right| < p \tag{4}$$

But from Theorem (a), we have:

$$\sum_{n=1}^{\infty} \psi(n,p,\delta) \frac{\{(1+B)n+(B-A)(1+p-\delta)\}p}{(B-A)} |a_{p+n}| \le p$$

The inequality (4) will be satisfied if:

$$\frac{(p+n)(c+p+n)}{(c+p)} |a_{p+n}| z^n$$

$$< \psi(n,p,\delta) \frac{\{(1+B)n+(B-A)(1+p-\delta)\}p}{(B-A)} |a_{p+n}|$$

for each n ∈ N , or if

$$|z| < \left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}(c+p)p\psi(n,p,\delta)}{(B-A)(p+n)(c+p+n)} \right]^{1/n}$$

for each n ∈ N

Hence f (z) is p-valent for $|z| < R$. To show the sharpness of the result, we take:

$$F(z) = z^p - \frac{(B-A)}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)} z^{p+n},$$

for each n ∈ N

Clearly F(z) belongs to the class H(A, B, p, δ) and thus:

$$f(z) = z^p - \frac{(c+p+n)(B-A)z^{p+n}}{(c+p)\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}$$

for each n ∈ N

Therefore,

$$\left| \frac{f'(z)}{z^{p-1}} - 1 \right| = \left| - \frac{(B-A)(c+p+n)(p+n)z^n}{\{(1+B)n+(B-A)(1+p-\delta)\}(c+p)\psi(n,p,\delta)} \right| = p$$

at z = R

Hence the result is sharp.

CONCLUSION

In this paper, we established a class preserving integral operator and a converse problem of this integral operator for a new class of analytic function.

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